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Boundary Values of Holomorphic Semigroups of Unbounded Operators and Similarity of Certain Perturbations

RHONDA J. HUGHES AND SHMUEL KANTOROVITZ*

Tufts University, Medford, Massachusetts 02155, Institute for Independent Study, Radcliffe College, Cambridge, Massachusetts 02138; University of Illinois at Chicago Circle, Chicago, Illinois 60680, and Bar-Ilan University, Ramat-Gan, Israel

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We obtain sufficient conditions for a "holomorphic" semigroup of unbounded operators to possess a boundary group of *bounded* operators. The theorem is applied to generalize to unbounded operators results of Kantorovitz about the similarity of certain perturbations. Our theory includes a result of Fisher on the Riemann–Liouville semigroup in $L^p(0, \infty)$ $1 < p < \infty$. In this particular case we give also an alternative approach, where the boundary group is obtained as the limit of *groups* in the weak operator topology.

1. INTRODUCTION

Kalisch [7] and Fisher [4] proved the existence of boundary values for the Riemann–Liouville semigroup $\{I^\zeta\}_{\zeta\in C^+}$ acting in $L^p(0, 1)$, $1 < p < \infty$, where for $f \in L^p(0, 1)$, $\zeta \in C^+$,

$$I^\zeta f(x) = \frac{1}{\Gamma(\zeta)} \int_0^x (x-t)^{\zeta-1} f(t)\, dt;$$

that is, for each $\eta \in R$ and $f \in L^p(0, 1)$, $\lim_{\xi\to 0+} I^{\xi+i\eta} f \equiv I^{i\eta} f$ exists in $L^p(0, 1)$, and $\{I^{i\eta}\}_{\eta\in R}$ is a strongly continuous group of bounded linear operators on $L^p(0, 1)$ such that

$$I^\zeta I^{i\eta} = I^{i\eta} I^\zeta = I^{\zeta+i\eta}, \qquad \text{for all } \zeta \in C^+,\ \eta \in R. \tag{1.1}$$

(The case $p = 2$ was first proved by Kober; cf. [5, p. 665].) The results in [4] are particularly interesting in light of the theory of semi-groups of unbounded operators developed in [6], for they establish the existence of a boundary group for the Riemann-Liouville integral acting in $L^p(0, \infty)$, where the operators I^ζ

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are unbounded. Thus there exists a strongly continuous group $\{I^{i\eta}\}_{\eta \in R}$ of bounded linear operators on $L^p(0, \infty)$ such that (1.1) holds, in the sense of unbounded operators.

Reference [4] therefore suggests a more general theory of "unbounded holomorphic semi-groups," especially with respect to the existence of boundary values. In section 2 of this paper, such a theory is developed: first, the concept of regular semi-group (cf. [9, Definition 2.1]) is modified to an unbounded setting, and in Theorem 2.2 conditions sufficient for the existence of a boundary group are given. The result is, basically, that regular semi-groups of unbounded operators (Definition 2.1) possess boundary groups.

Although of interest in its own right, our main objective in extending the boundary value theory to holomorphic semi-groups of unbounded operators is the application to problems concerning similarity of certain perturbations in $L^p(0, \infty)$. Kantorovitz [8] has proved that in the abstract case of certain pairs of bounded operators satisfying the commutation relation $[M, J] = AJ^2$, where A is a bounded operator commuting with both M and J, $T_\alpha = M + \alpha A J$ and $T_\beta = M + \beta A J$ are similar if $\operatorname{Re} \alpha = \operatorname{Re} \beta$ and only if $| \operatorname{Re} \alpha | = | \operatorname{Re} \beta |$, for $\alpha, \beta \in C$. The motivating example is the case where $Mf(x) = xf(x)$ and $Jf(x) = \int_0^x f(t)\, dt$ on $L^p(0, 1)$ (in which case the similarity holds if and only if $\operatorname{Re} \alpha = \operatorname{Re} \beta$). In [9] Kantorovitz and Pei extended the above theory to the case where M is unbounded, and J is bounded (for example, if M is the multiplication operator in $L^p(0, \infty)$, and J the "weighted Volterra operator" W defined by

$$Wf(x) = \int_0^x e^{t-x} f(t)\, dt.$$

It is natural to ask whether some such similarity result holds when both M and J are allowed to be unbounded operators; this situation occurs, for example, when M and J are the multiplication and Volterra operators, respectively, acting in $L^p(0, \infty)$.

A crucial element in the proofs in both the bounded and "semi-bounded" case is the assumption that the operator J may be embedded in a holomorphic semi-group of bounded operators possessing a boundary group on the imaginary axis. Indeed, the similarity of the perturbations T_α and T_β is implemented by appropriate operators belonging to the boundary group.

Similarity results for perturbations of certain pairs M and J of unbounded operators are obtained in section 3. In effect, we prove that if M is a closed operator and if $J = J(1)$, where $\{J(\zeta)\}_{\zeta \in C^+}$ is a regular semi-group of unbounded operators, and M and J satisfy certain technical conditions, then T_α is similar to T_β if $\operatorname{Re} \alpha = \operatorname{Re} \beta$ and $D(M) \subset D(J)$ (see Theorems 3.3 and 3.4).

A considerable portion of this paper is devoted to applications (Section 4). For instance, if $f \in L^p(0, \infty)$, $1 < p < \infty$, $Mf(x) = xf(x)$, $Jf(x) = \int_x^\infty f(t)\, dt$, and M and J have maximal domains in $L^p(0, \infty)$, then an application of Theorems 3.3 and 3.4 yields

THEOREM 4.10. *If* $\alpha, \beta \in C$ *and* $\operatorname{Re}\alpha = \operatorname{Re}\beta$, *then* $M + \alpha J$ *is similar to* $M + \beta J$. *Moreover, the similarity is implemented by* $J^{i\,\operatorname{Im}(\alpha-\beta)}$, *where* $\{J^{i\eta}\}_{\eta\in R}$ *is the boundary group provided by Theorem* 2.2.

Analogous results are obtained for the Volterra operator (Theorem 4.6).

Also, in the course of applying our results to the examples in Section 4, we give a new proof that the Riemann-Liouville semi-group acting in $L^p(0, N)$, $N > 0$, $1 < p < \infty$, possesses a boundary group (see Theorem 4.4 and following remark).

In Section 5 we present an alternate approach to the boundary value problem in the case of the Riemann-Liouville semi-group on $L^p(0, \infty)$. We utilize many of the results of Section 4 to construct (without appealing to Theorem 2.2) the boundary group $\{I^{i\eta}\}_{\eta\in R}$ as a limit of groups $\{W_\epsilon^{i\eta}\}_{\eta\in R}$ (as $\epsilon \to 0^+$) in the weak operator topology.[1]

We shall use standard notation (for example, cf. [5]), except that I^1 will denote the Volterra operator, and **1** the identity operator.

2. Regular Semi-groups of Unbounded Operators

Let X be a Banach space, and $\{P_N\}_{N\in Z^+}$ a family of projections on X such that

(i) $\| P_N \| \leqslant M$, where M is a constant independent of $N \in Z^+$, and

(ii) $\| P_N x - x \| \to 0$ as $N \to \infty$, for each $x \in X$.

(We point out that an arbitrary directed set could serve as the index set in this development.) For each $N \in Z^+$, let $\{J_N(\zeta)\}_{\zeta\in C^+}$ be a regular semi-group of bounded linear operators on the Banach space $P_N X$ (cf. [9, Definition 2.1]); let $\{J(\zeta)\}_{\zeta\in C^+}$ be a one-parameter family of (possibly unbounded) linear operators in X such that for each $\zeta \in C^+$,

$$\text{Domain } (J(\zeta)) = \{x \in X \mid \lim_{N\to\infty} J_N(\zeta)x \text{ exists in } X\},$$

and for each $x \in D(J(\zeta))$, $J(\zeta)x = \lim_{N\to\infty} J_N(\zeta)x$. (For $x \in X$, $J_N(\zeta)x$ is to be interpreted as $J_N(\zeta)\, P_N x$.) In the terminology of Trotter [13], $\{P_N X\}_{N\in Z^+}$ is a sequence of Banach spaces approximating X, and for each $\zeta \in C^+$, $J(\zeta)$ is the limit of the sequence of operators $(J_N(\zeta))_{N\in Z^+}$. We shall refer to $\{J_N(\zeta)\}_{\zeta\in C^+}$, $N \in Z^+$, as *an approximating sequence of semi-groups.*

DEFINITION 2.1. A one-parameter family $\{J(\zeta)\}_{\zeta\in C^+}$ of densely-defined linear operators in X with an approximating sequence of regular semi-groups $\{J_N(\zeta)\}_{\zeta\in C^+}$, $N \in Z^+$, is said to be a *regular semi-group of operators* if the following hold:

[1] Using different methods, M. J. Fisher, "Purely Imaginary Powers of Certain Differential Operators, I," *Amer. J. Math.* **93** (1971), 452–478; has obtained the boundary group in Theorem 5.1 as a limit in the strong operator topology.

(i) if $\{J_N(i\eta)\}_{\eta\in R}$ is the strongly continuous group of boundary values for $\{J_N(\zeta)\}_{\zeta\in C^+}$, then there exist constants K and ν, independent of $N \in Z^+$, such that

$$\| J_N(i\eta)\| \leqslant Ke^{\nu|\eta|}, \ \eta \in R, \ N \in Z^+; \tag{2.1.1}$$

(ii) if

$$\tilde{D} = \left\{ x \in \bigcap_{\alpha,\beta\in C^+} D(J(\alpha)\, J(\beta)) \,\middle|\, \begin{array}{l} J_s J_t x = J_{s+t}x, \ s, t > 0, \\ J_t x \text{ strongly continuous, } t > 0, \\ J_t x \to x \text{ as } t \to 0^+, \end{array} \right\}, \tag{2.1.2}$$

then $\tilde{D}$ is dense in X.

Note. If the operators $J(\zeta)$ are bounded on X, then our definition of regularity is easily seen to coincide with Definition 2.1 of [9]. We wish to emphasize that the latter definition of regularity includes the following condition: if $\gamma_N(s)$ is the Nörland function of $\{J_N(\zeta)\}_{\zeta\in C^+}$, and $(\alpha_{0,N}, \alpha_{1,N})$ is the largest interval such that the equation $\gamma_N(s) = \pi/2\alpha$ has a (necessarily unique) solution $s_N = s_{0,N}(\alpha)$ when $0 \leqslant \alpha_{0,N} < \alpha < \alpha_{1,N} \leqslant \infty$, then $\alpha_{1,N} > 1$.

In light of the fact that regular semi-groups of bounded linear operators possess boundary values on the imaginary axis, the following result provides some justification for the terminology of Definition 2.1.

THEOREM 2.2. *Let $\{J(\zeta)\}_{\zeta\in C^+}$ be a regular semi-group of unbounded linear operators in X. Then there exists a strongly continuous group of bounded linear operators $\{J(i\eta)\}_{\eta\in R}$ on X such that*

(i) $J(i\eta)x = \lim_{N\to\infty} J_N(i\eta)x$, *all* $x \in X$, $\eta \in R$;

(ii) $\| J(i\eta)\| \leqslant Ke^{\nu|\eta|}$, *where K and ν are the constants in* (2.1.1);

(iii) $J(i\eta)\, J(\alpha) = J(\alpha + i\eta)$, *as unbounded operators in X, for each $\alpha > 0$, $\eta \in R$;*

(iv) *if*

$$D = \left\{ x \in \bigcap_{s,t>0} D(J(s)\, J(t)) \,\middle|\, \begin{array}{l} J_s J_t x = J_{s+t}x, \\ J_t x \text{ strongly continuous, } t > 0, \\ J_t x \to x \text{ as } t \to 0^+, \end{array} \right\}, \tag{2.2.1}$$

then for each $x \in D$, $\eta \in R$,

$$J(i\eta)x = \lim_{\xi\to 0^+} J(\xi + i\eta)x. \tag{2.2.2}$$

Proof. If $x \in \tilde{D}$, then for each $\alpha > 0$, $J(\alpha)x \in \tilde{D}$. Thus $\bigcup_{\alpha>0} J(\alpha)\tilde{D}$ is dense in X, for if $x \in \tilde{D}$ then $J(\alpha)x \to x$ as $\alpha \to 0^+$. Now fix $x \in \tilde{D}$, $\alpha > 0$ and $\eta \in R$. Then

$$\begin{aligned} &\| J_N(i\eta)\, J(\alpha)x - J(\alpha + i\eta)x \| \\ &\quad \leqslant \| J_N(i\eta)\, J(\alpha)x - J_N(i\eta)\, J_N(\alpha)x \| + \| J_N(\alpha + i\eta)x - J(\alpha + i\eta)x \| \quad (2.2.3) \\ &\quad \to 0 \qquad \text{as} \quad N \to \infty, \end{aligned}$$

since $\| J_N(i\eta)\| \leqslant Ke^{\nu|\eta|}$, and $x \in \tilde{D}$. Since $\bigcup_{\alpha>0} J(\alpha)\tilde{D}$ is dense in X, it follows from (2.2.3) that $\lim_{N\to\infty} J_N(i\eta)x$ exists for each $x \in X$, and defines a bounded linear operator which we shall denote by $J(i\eta)$. Moreover, $\| J(i\eta)\| \leqslant Ke^{\nu|\eta|}$, and so (i) and (ii) are proved.

In order to show that $\{J(i\eta)\}_{\eta\in R}$ is a strongly continuous group of operators on X, fix $\eta, \gamma \in R$. For each $N \in Z^+$, $J_N(i\eta)\, J_N(i\gamma) = J_N(i(\eta + \gamma))$, so for $x \in X$,

$$\begin{aligned} &\| J_N(i\eta)\, J(i\gamma)x - J(i(\eta + \gamma))x \| \\ &\quad \leqslant \| J_N(i\eta)\, J(i\gamma)x - J_N(i\gamma)\, J_N(i\eta)x \| + \| J_N(i(\gamma + \eta))x - J(i(\gamma + \eta))x \| \\ &\quad \leqslant Ke^{\nu|\eta|} \| J(i\gamma)x - J_N(i\gamma)x \| + \| J_N(i(\gamma + \eta))x - J(i(\gamma + \eta))x \| \\ &\quad \to 0 \qquad \text{as} \quad N \to \infty. \end{aligned}$$

Therefore $J(i\eta)\, J(i\gamma)x = J(i(\eta + \gamma))x$. Since η, γ and x were chosen arbitrarily, and $J(0) = 1$, the identity operator on X, it follows that $\{J(i\eta)\}_{\eta\in R}$ is a group of bounded operators on X.

Now for each $N \in Z^+$, $\phi_N : \eta \to J_N(i\eta)$, $\eta \geqslant 0$, is strongly continuous, so in particular it is strongly measurable. Since the strong limit of a sequence of strongly measurable functions is strongly measurable (cf. [5, Theorem 3.5.4]), $\phi: \eta \to J(i\eta)$, $\eta \geqslant 0$, is strongly measurable. By Theorem 10.2.3 of [5], ϕ is strongly continuous. For $0 \leqslant \eta \leqslant 1$, $\| J(i\eta)\| \leqslant B$, for some constant $B > 0$. Because $J(i\gamma)$ maps onto X for each $\gamma \in R$, it follows that $J(i\eta)x \to x$ as $\eta \to 0^+$, for each $x \in X$ (similarly for $\eta \to 0^-$), and so $\{J(i\eta)\}_{\eta\in R}$ is a strongly continuous group.

Now suppose $x \in D(J(\alpha))$, with $\alpha > 0$. Then for $\eta \in R$,

$$\begin{aligned} &\| J_N(\alpha + i\eta)x - J(i\eta)\, J(\alpha)x \| \\ &\quad \leqslant \| J_N(\alpha + i\eta)x - J_N(i\eta)\, J(\alpha)x \| + \| J_N(i\eta)\, J(\alpha)x - J(i\eta)\, J(\alpha)x \| \\ &\quad \to 0 \qquad \text{as} \quad N \to \infty, \end{aligned}$$

so that $x \in D(J(\alpha + i\eta))$, and $J(\alpha + i\eta)x = J(i\eta)J(\alpha)x$. Thus

$$J(i\eta)\, J(\alpha) \subset J(\alpha + i\eta), \qquad \alpha > 0, \eta \in R. \tag{2.2.4}$$

Replacing α by $\alpha - i\eta$, and applying $J(-i\eta)$ to both sides of (2.2.4), we obtain (iii); (iv) follows immediately, for the operators $J(i\eta)$ are continuous.

As an immediate consequence of (iii) we have

COROLLARY 2.3. $D(J(\alpha)) = D(J(\operatorname{Re} \alpha))$, $\alpha \in C^+$.

Remark. We see from the above corollary that $\tilde{D} = D$, so that by (iv) of Theorem 2.2 the bounded operator $J(i\eta)$ is uniquely determined by the values of $\lim_{\xi\to 0^+} J(\xi + i\eta)x$ for $x \in D$.

The following proof is a generalization of the argument in [6, Theorem 4.4].

COROLLARY 2.4. *For* $\alpha, \beta \in C^+$, *with* $0 < \operatorname{Re} \alpha < \operatorname{Re} \beta$,

$$D(J(\beta)) \subseteq D(J(\alpha)).$$

Proof. Suppose $x \in D(J(\beta))$. Fix $N, M \in Z^+$ and define, for $\zeta \in C^+$,

$$\Phi_{N,M}(\zeta) = \exp(\nu\zeta^2)(J_N(\zeta)x - J_M(\zeta)x),$$

where ν is the constant in (2.1.1). This X-valued function is strongly continuous for $0 \leqslant \operatorname{Re} \zeta \leqslant \operatorname{Re} \beta$, and holomorphic for $0 < \operatorname{Re} \zeta \leqslant \operatorname{Re} \beta$. Set $\beta_0 = \operatorname{Re} \beta$, and let $\zeta = \xi + i\eta$, where $0 \leqslant \xi < \beta_0$. Then

$$\begin{aligned} \| J_N(\zeta)x - J_M(\zeta)x \| &= \| J_N(i\eta)\, J_N(\xi)x - J_M(i\eta)\, J_N(\xi)x \| \\ &\leqslant K e^{\nu|\eta|}(\| J_N(\xi)x \| + \| J_M(\xi)x \|). \end{aligned}$$

Thus

$$\| \Phi_{N,M}(\zeta)\| \leqslant K \exp(\xi^2 + \tfrac{1}{4})(\| J_N(\xi)x \| + \| J_M(\xi)x \|),$$

so $\| \Phi_{N,M}(\xi)\|$ is bounded in the strip $0 \leqslant \xi \leqslant \beta_0$. By the three lines theorem (cf. [2, Theorem VI.10.3]), we see that if $\alpha \in C$ with $0 < \operatorname{Re} \alpha < \operatorname{Re} \beta$ and $0 \leqslant r \leqslant 1$, then (using Corollary 2.3)

$$\begin{aligned} \| \Phi_{N,M}(\alpha)\| &\leqslant K \exp(\nu(\beta_0{}^2 + \tfrac{1}{4}))\, \| P_N x - P_M x \|^r \, \| J_N(\beta_0)x - J_M(\beta_0)x \|^{1-r} \\ &\to 0 \qquad \text{as} \quad N, M \to \infty. \end{aligned}$$

Therefore $\{J_N(\alpha)x\}_{N \in Z^+}$ is a Cauchy sequence in X, so that $x \in D(J(\alpha))$.

A suitable modification of the proof of Theorem 6 in [4] yields

COROLLARY 2.5. *The spectral radius of* $J(i\eta)$ *satisfies* $r(J(i\eta)) \leqslant e^{\nu|\eta|}$, *and*

$$\sigma(J(i\eta)) \subseteq \{\lambda \in C \mid e^{-\nu|\eta|} \leqslant |\lambda| \leqslant e^{\nu|\eta|}\}.$$

3. SIMILARITY

Let $J = J(1)$, where $\{J(\zeta)\}_{\zeta \in C^+}$ is a regular semi-group of unbounded operators in X, with approximating semi-groups $\{J_N(\zeta)\}_{\zeta \in C^+}$, $N \in Z^+$; set $J_N = J_N(1)$. Let M be a closed operator in X with domain $D(M)$, and suppose that for each $N \in Z^+$ the following hold:

(i) A is a non-zero bounded operator on X commuting with M and J_N;

(ii) $J_N(s + it)\, D(M) \subset D(M)$ for $s + it$ in some rectangle $0 \leqslant s \leqslant a$, $|t| \leqslant a$ (a may depend on N); (3.0.1)

(iii) J_N is M-Volterra with respect to A (cf. [9, Definition 1.1]).

We now define, for $\alpha \in C$,

$$T_\alpha = M + \alpha A J,$$

with domain $D(T_\alpha) = D(M) \cap D(J)$ (except, of course, when $\alpha = 0$). For each $N \in Z^+$, let

$$T_{\alpha,N} = M + \alpha A J_N ,$$

with domain $D(M)$. $T_{\alpha,N}$ is a closed operator in X, and it is easy to check that despite the change in the standing hypotheses in [9, p. 1183] (we assume that $\{J_N(\zeta)\}_{\zeta \in C^+}$ is of class (C_0) on $P_N X$, rather than on X), Theorem 2.5 of [9] remains valid. We omit the proof of the following; it is exactly as in [9].[2]

THEOREM 3.1. *The following equivalent identities are valid for all* $\zeta \in \bar{C}^+$, *and for each* $N \in Z^+$:

(i) $[J_N(\zeta), M] \subset \zeta A J_N(\zeta + 1)$,

(ii) $J_N(\zeta) M \subset T_{\zeta,N} J_N(\zeta)$,

(iii) $J_N(\zeta) T_{-\zeta,N} \subset M J_N(\zeta)$.

Remark. We obtain from Theorem 3.1 that

$$J_N(i\eta)M = (M + i\eta A J_N)\, J_N(i\eta), \qquad \text{on } D(M), \tag{3.1.1}$$

but M and $M + i\eta A J_N$ are not similar, for $J_N(i\eta)$ is not invertible (as an operator on X).

We shall need the following simple result:

LEMMA 3.2. *A commutes with $J(\zeta)$, for each $\zeta \in \bar{C}^+$.*

Proof. Since A commutes with J_N for each $N \in Z^+$, it follows as in [9, Lemma 2.3] that A commutes with $J_N(\zeta)$ for all $\zeta \in \bar{C}^+$. Now if $x \in D(J(\zeta))$ ($=X$ if $\operatorname{Re} \zeta = 0$), then $AJ(\zeta)x = \lim_{N\to\infty} AJ_N(\zeta)x = \lim_{N\to\infty} J_N(\zeta)\, Ax$, so $Ax \in D(J(\zeta))$ and $AJ(\zeta)x = J(\zeta)\, Ax$.

THEOREM 3.3. *For* $\eta \in R$,

$$J(i\eta)\, MJ(-i\eta) \supset T_{i\eta} .$$

In particular, if $D(M) \subset D(J)$, then M and $T_{i\eta}$ are similar, with $J(i\eta)$ implementing the similarity.

Proof. Fix $\eta \in R$, and let $x \in D(T_{i\eta})$. Then by (3.1.1),

$$J_N(i\eta)\, Mx = MJ_N(i\eta)x + i\eta A J_N(1 + i\eta)x.$$

[2] A small gap in the proof of Theorem 2.5 of [9] # is filled in a forthcoming paper by the authors, tentatively titled "Unbounded Derivations and Similarity of Closed Operators."

But $J_N(i\eta) Mx \to J(i\eta) Mx$ and $J_N(1 + i\eta)x \to J(1 + i\eta)x$, since $x \in D(J) = D(J(1 + i\eta))$ (Corollary 2.3). Thus $MJ_N(i\eta)x \to J(i\eta) Mx - i\eta AJ(1 + i\eta)x$. Since $J_N(i\eta)x \to J(i\eta)x$, and M is closed, we have that $J(i\eta)x \in D(M)$ and

$$\begin{aligned} J(i\eta) Mx &= MJ(i\eta)x + i\eta AJ(1 + i\eta)x \\ &= MJ(i\eta)x + i\eta AJ(i\eta) Jx. \end{aligned}$$

By Lemma 3.2, $J(i\eta)(M - i\eta AJ)x = MJ(i\eta)x$, so

$$(M - i\eta AJ)x = J(-i\eta) MJ(i\eta)x.$$

Replacing η by $-\eta$, we obtain the desired result; the last statement is clear.

THEOREM 3.4. *If* $\alpha, \beta \in C\backslash\{0\}$, *and* $\operatorname{Re}\alpha = \operatorname{Re}\beta$, *then* T_α *is similar to* T_β. *In particular*,

$$J(i \operatorname{Im}(\alpha - \beta))T_\beta = T_\alpha J(-i \operatorname{Im}(\alpha - \beta)). \tag{3.4.1}$$

Proof. We will first show that for each $\eta \in R$, and $x \in D(M) \cap D(J)$,

$$AJ \cdot J(i\eta)x = AJ(1 + i\eta)x. \tag{3.4.2}$$

Indeed, by Theorem 3.3 $J(i\eta) MJ(-i\eta)x = (M + i\eta AJ)x$ for $x \in D(M) \cap D(J)(=D(T_{i\eta}))$. Thus

$$\begin{aligned} J(i\eta) Mx &= (M + i\eta AJ) J(i\eta)x \\ &= MJ(i\eta)x + i\eta AJ \cdot J(i\eta)x. \end{aligned} \tag{3.4.3}$$

On the other hand,

$$J(i\eta) Mx = MJ(i\eta)x + i\eta AJ(i\eta) Jx, \tag{3.4.4}$$

and (3.4.2) now follows from (3.4.3) and (3.4.4).

Combining (3.4.2) with Theorem 3.3 we have, on $D(M) \cap D(J)$,

$$J(i \operatorname{Im}(\alpha - \beta))(M + i \operatorname{Im}\beta AJ) = (M + i \operatorname{Im}\alpha AJ) J(i \operatorname{Im}(\alpha - \beta)), \tag{3.4.5}$$

and we obtain (3.4.1) by adding

$$\operatorname{Re}\alpha J(i \operatorname{Im}(\alpha - \beta)) AJ = \operatorname{Re}\alpha AJ \; J(i \operatorname{Im}(\alpha - \beta))$$

to both sides of (3.4.5).

Note. We have not proved that $J(\alpha) J(i\eta) = J(i\eta) J(\alpha)$ in this abstract setting, but we shall do so in certain concrete examples (see Section 4).

4. Applications

Consider the one-parameter family of operators $\{I^\zeta\}_{\zeta\in C^+}$, where for $f\in L^p(0,\infty)$, $1<p<\infty$,

$$I^\zeta f(x) = \frac{1}{\Gamma(\zeta)}\int_0^x (x-t)^{\zeta-1} f(t)\,dt,$$

the usual Riemann-Liouville semi-group. If I^ζ has domain $D(I^\zeta) = \{f\in L^p(0,\infty) \mid I^\zeta f\in L^p(0,\infty)\}$, then I^ζ is a closed, densely defined linear operator in $L^p(0,\infty)$ (cf. [6, Proposition 4.1]). Let M denote the multiplication operator $Mf(x) = xf(x)$ acting in $L^p(0,\infty)$, with domain $D(M) = \{f\in L^p(0,\infty) \mid Mf\in L^p(0,\infty)\}$, and for $\alpha\in C$, set $T_\alpha = M + \alpha I^1$. We shall apply the theorems of the preceding sections in order to obtain similarity results, corresponding to those in [8, 9], for the unbounded operators T_α.

For each $N\in Z^+$, and $f\in L^p(0,\infty)$, define

$$\begin{aligned} P_N f(x) &= f(x), \qquad 0\leqslant x\leqslant N,\\ &= 0, \qquad\qquad x>N. \end{aligned}$$

If we set $I_N{}^\zeta f(x) = P_N I^\zeta P_N f(x)$ $(=P_N I^\zeta f(x))$, then clearly

$$D(I^\zeta) = \{f\in L^p(0,\infty) \mid \lim_{N\to\infty} I_N{}^\zeta f \text{ exists in } L^p(0,\infty)\}.$$

To apply Theorems 3.3 and 3.4, we must prove that $\{I^\zeta\}_{\zeta\in C^+}$ is a regular semi-group of unbounded operators in $L^p(0,\infty)$, with approximating semi-groups $\{I_N{}^\zeta\}_{\zeta\in C^+}$, $N\in Z^+$, and also that (3.0.1) holds, with $Af(x) = -f(x)$ on $L^p(0,\infty)$. The verifications of (i)–(iii) in (3.0.1) are trivial, so we turn to the proof of the regularity of $\{I^\zeta\}_{\zeta\in C^+}$.

For $N\in Z^+$, let ν_N be the type of $\{I_N{}^\zeta\}_{\zeta\in C^+}$. It is well known that $\{I_N{}^\zeta\}_{\zeta\in C^+}$ is a holomorphic semi-group of class (C_0) on $L^p(0,N)$, so we need only show that $\nu_N<\infty$ and $\alpha_{1,N}>1$. Kalisch [7] and Fisher [4] have proved that $\nu_N<\infty$, and $\alpha_{1,N}>1$ may be proved exactly as in [9, p. 1195], since $\| I_N^{\xi+i\eta}\| \leqslant N^\xi/\xi \mid \Gamma(\xi+i\eta)|$ implies $\gamma_N(\xi)\leqslant \pi/2$ for all $\xi>0$.

We remark that Fisher's proof, which depends on a theorem of Muckenhoupt (cf. [1, Theorem 1]), gives (2.1.1) and also supplies the boundary group we obtain in Theorem 2.2; the bounds obtained in [7] are unbounded with respect to N. Using the Mihlin multiplier theorem (cf. [7]), we now give a new proof of (2.1.1), independent of the techniques used in [4].

We first introduce the weighted Riemann-Liouville semi-groups, which will be useful in our next example as well. For $\epsilon>0$, $\zeta\in C^+$ and $f\in L^p(0,\infty)$, $1<p<\infty$, let

$$W_\epsilon{}^\zeta f(x) = \frac{1}{\Gamma(\zeta)}\int_0^x e^{\epsilon(t-x)}(x-t)^{\zeta-1} f(t)\,dt.$$

LEMMA 4.1. *For each* $\epsilon > 0$ *and* $\zeta \in C^+$, W_ϵ^ζ *is a bounded operator on* $L^p(0, \infty)$, $1 \leqslant p \leqslant \infty$, *and for* $1 < p < \infty$,

$$\| W_\epsilon^\zeta \| \leqslant e^{\pi|\eta|/2} \epsilon^{-\xi} \max\{|\zeta|, 1\}, \qquad \zeta = \xi + i\eta. \tag{4.1.1}$$

Proof. Let

$$\begin{aligned} K_\epsilon^\zeta(x) &= \frac{1}{\Gamma(\zeta)} e^{-\epsilon x} x^{\zeta-1}, && x > 0, \\ &= 0, && x \leqslant 0. \end{aligned}$$

and

$$\begin{aligned} f_0(x) &= f(x), && x > 0, \\ &= 0, && x \leqslant 0, \end{aligned}$$

for any function f defined a.e. on $(0, \infty)$. Clearly

$$(W_\epsilon^\zeta f)_0 = K_\epsilon^\zeta * f_0 ,$$

and it follows that

$$\| W_\epsilon^\zeta \| \leqslant \| K_\epsilon^\zeta \|_{L^1(R)} = \frac{\Gamma(\xi)}{|\Gamma(\zeta)|} \epsilon^{-\xi} < \infty \tag{4.1.2}$$

is valid on $L^p(0, \infty)$ for $1 \leqslant p \leqslant \infty$.

For $1 < p < \infty$, we wish to obtain the estimate (4.1.1). For $h \in \mathscr{S}$, the Schwartz space, denote by $\hat{h}$ the Fourier transform of h. Then

$$(W_\epsilon^\zeta f)(x) = (\hat{K}_\epsilon^\zeta \hat{f}_0)\hat{}(-x), \qquad f \in \mathscr{S}.$$

We have (cf. [3, p. 12])

$$\begin{aligned} \hat{K}_\epsilon^\zeta(y) &= \frac{1}{\Gamma(\zeta)} \int_0^\infty e^{-iyx - \epsilon x} x^{\zeta-1} \, dx \\ &= (\epsilon^2 + y^2)^{-\zeta/2} e^{-i\zeta \arctan(y/\epsilon)}. \end{aligned}$$

Differentiating, we obtain

$$y \frac{d}{dy} \hat{K}_\epsilon^\zeta(y) = -i\zeta(\epsilon^2 + y^2)^{-\zeta/2} \frac{y}{\epsilon + iy} e^{-i\zeta \arctan(y/\epsilon)},$$

so that for all $y \in R$,

$$|\hat{K}_\epsilon^\zeta(y)| \leqslant \epsilon^{-\xi} e^{\pi|\eta|/2}, \qquad \zeta = \xi + i\eta, \quad \xi > 0,$$

and

$$\left| y \frac{d}{dy} \hat{K}_\epsilon^\zeta(y) \right| \leqslant |\zeta| \epsilon^{-\xi} e^{\pi|\eta|/2};$$

thus (4.1.1) now follows from the Mihlin multiplier theorem.

THEOREM 4.2. *For each* $\epsilon > 0$ *and* $1 \leqslant p < \infty$, $\{W_\epsilon^\zeta\}_{\zeta \in C^+}$ *is a holomorphic semi-group of class* (C_0) *on* $L^p(0, \infty)$. *If* $1 < p < \infty$, $\{W_\epsilon^\zeta\}_{\zeta \in C^+}$ *is a regular semi-group of bounded operators on* $L^p(0, \infty)$ *with boundary group* $\{W_\epsilon^{i\eta}\}_{\eta \in R}$. *Moreover, uniformly in* $\epsilon > 0$,

$$\| W_\epsilon^{i\eta} \| \leqslant e^{\pi|\eta|/2}, \qquad \eta \in R. \tag{4.2.1}$$

Proof. For $1 \leqslant p < \infty$ and fixed $\epsilon > 0$, the semi-group property follows in the usual manner from Fubini's theorem, and the (C_0) property may be proved by noting that $\epsilon^\xi K_\epsilon^\xi$, for $\xi > 0$, is an approximate identity in $L^1(R)$. Next, we show that W_ϵ^ζ is a strongly continuous function of $\zeta \in C^+$. Indeed, for ζ, $\omega \in C^+$, $N > 0$ and $f \in L^p(0, \infty)$,

$$\begin{aligned} \| W_\epsilon^\zeta f - W_\epsilon^\omega f \|^p \leqslant & \int_0^N | I^\zeta(e^{\epsilon t} f) - I^\omega(e^{\epsilon t} f)|^p \, dx \\ & + \int_N^\infty e^{-xp\epsilon/2} \left| \frac{1}{\Gamma(\zeta)} \int_0^x (x-t)^{\zeta-1} e^{\epsilon(t-x)/2} e^{\epsilon t/2} f(t) \, dt \right|^p dx \\ & + \int_N^\infty e^{-xp\epsilon/2} \left| \frac{1}{\Gamma(\omega)} \int_0^x (x-t)^{\omega-1} e^{\epsilon(t-x)/2} e^{\epsilon t/2} f(t) \, dt \right|^p dx, \end{aligned}$$

where I^ζ is the usual Riemann-Liouville operator. Now set $E = \{f \text{ measurable on } (0, \infty) \mid e^{\epsilon t/2} f(t) \in L^p(0, \infty)\}$. If $f \in E$, then by (4.1.2)

$$\begin{aligned} \| W_\epsilon^\zeta f - W_\epsilon^\omega f \|^p \leqslant & \int_0^N | I^\zeta(e^{\epsilon t} f) - I^\omega(e^{\epsilon t} f)|^p \, dx \\ & + F(\zeta, \omega) \, e^{-N\epsilon P/2} \| e^{\epsilon t/2} f \|^P, \end{aligned}$$

where

$$F(\zeta, \omega) = \left[\frac{\Gamma(\xi)}{|\Gamma(\zeta)|} \left(\frac{\epsilon}{2}\right)^{-\xi} + \frac{\Gamma(\sigma)}{|\Gamma(\omega)|} \left(\frac{\epsilon}{2}\right)^{-\sigma} \right]^P$$

and $\xi = \operatorname{Re} \zeta$, $\sigma = \operatorname{Re} \omega$. For $\zeta \in C^+$ fixed, and ω belonging to some closed disc centered at ζ and lying in C^+, $F(\zeta, \omega) \leqslant M$ for some constant $M > 0$. Thus given $\delta > 0$, $F(\zeta, \omega) \, e^{-N\epsilon P/2} \| e^{\epsilon t/2} f \|^P < \delta^P$ for a suitable $N > 0$. For this N, we apply the strong continuity of $\{I^\zeta\}$ in $L^P(0, N)$ to obtain

$$\limsup_{\omega \to \zeta} \| W_\epsilon^\zeta f - W_\epsilon^\omega f \| \leqslant \delta.$$

Since E is dense in $L^P(0, \infty)$ and $\| W_\epsilon^\zeta \|$ is bounded on compact subsets of C^+, the strong continuity of $\{W_\epsilon^\zeta\}_{\zeta \in C^+}$ is proved. The fact that $\{W_\epsilon^\zeta\}$ is holomorphic in C^+ now follows in a straightforward manner from Morera's theorem.

If $1 < p < \infty$, then Lemma 4.1 gives (with $Q = \{\xi + i\eta \in C \mid 0 < \xi \leqslant 1, |\eta| \leqslant 1\}$),

$$\sup_Q \| W_\epsilon^\zeta \| \leqslant 2^{1/2} e^{\pi/2} \max(\epsilon^{-1}, 1). \tag{4.2.2}$$

By [5, Theorem 17.9.1], it follows that $\{W_\epsilon^\zeta\}_{\zeta\in C^+}$ possesses a strongly continuous boundary group $\{W_\epsilon^{i\eta}\}_{\eta\in R}$, given by $W_\epsilon^{i\eta} = \lim_{\xi\to 0+} W_\epsilon^{\xi+i\eta}$, in the strong operator topology. For each $f \in L^p(0, \infty)$, it follows from Lemma 4.1 that

$$\| W_\epsilon^{i\eta} f \| \leqslant e^{\pi|\eta|/2} \| f \|, \qquad \text{for} \quad |\eta| \leqslant 1,$$

and a standard argument yields (4.2.1). According to [9, Definition 2.1], we must also show that $\alpha_1 > 1$. But (4.2.2) implies that $\gamma(\xi) \leqslant \pi/2$, from which we conclude that $\alpha_0 \geqslant 1$.

By applying (4.1.2) (with $\zeta = \xi > 0$) and (4.2.1), we obtain the following:

COROLLARY 4.3. *For all $\epsilon > 0$, and $\zeta \in C^+$ $(\zeta = \xi + i\eta)$,*

$$\| W_\epsilon^\zeta \| \leqslant \epsilon^{-\zeta} e^{\pi|\eta|/2}. \tag{4.3.1}$$

Proof. Apply (4.1.2) with $\zeta = \xi$ and (4.2.1).

THEOREM 4.4. *For each $N \in Z^+$, and $\eta \in R$,*

$$\| I_N^{i\eta} \|_{L^p(0,N)} \leqslant e^{\pi|\eta|/2} \qquad (1 < p < \infty), \tag{4.4.1}$$

where $\{I_N^{i\eta}\}_{\eta\in R}$ is the boundary group of $\{I_N^\zeta\}_{\zeta\in C^+}$.

Proof. Fix $N \in Z^+$, $\epsilon > 0$, and let $I_{\epsilon,N}^\zeta$, $\zeta \in C^+$, *denote* $P_N W_\epsilon^\zeta P_N$. Then $I_{\epsilon,N}^\zeta$ is a bounded linear operator on $L^p(0, N)$, and since $\| I_{\epsilon,N}^\zeta \|_{L^p(0,N)} \leqslant \| W_\epsilon^\zeta \|_{L^p(0,\infty)}$, one sees that $\{I_{\epsilon,N}^\zeta\}_{\zeta\in C^+}$ is a regular semi-group on $L^p(0, N)$. If $\{I_{\epsilon,N}^{i\eta}\}_{\eta\in R}$ is the boundary group, then by Theorem 4.2,

$$\| I_{\epsilon,N}^{i\eta} \| = \lim_{\xi\to 0^+} \| I_{\epsilon,N}^{\xi+i\eta} \| \leqslant \lim_{\xi\to 0^+} \| W_\epsilon^{\xi+i\eta} \| = \| W_\epsilon^{i\eta} \| \leqslant e^{\pi|\eta|/2}, \tag{4.4.2}$$

for each $\eta \in R$.

In order to verify (4.4.1), we will show that $I_N^{i\eta} = \lim_{\epsilon\to 0+} I_{\epsilon,N}^{i\eta}$ in the strong operator topology; in light of (4.4.2), it suffices to prove this for elements belonging to a dense subset of $L^p(0, N)$.

First, note that $I_N^\zeta = \lim_{\epsilon\to 0+} I_{\epsilon,N}^\zeta$ in the strong topology, since for $f \in L^p(0, N)$, $I_{\epsilon,N}^\zeta f(x) \to I_N^\zeta f(x)$ pointwise a.e., and the dominated convergence theorem applies. Now let $f \in \text{Range}(I_N^1)$, which is a dense subset of $L^p(0, N)$ (cf. [9, Lemma 2.2]). If $f = I_N^1 g$, where $g \in L^p(0, N)$, and $\delta > 0$, then

$$\begin{aligned} \| I_N^{i\eta} f - I_{\epsilon,N}^{i\eta} f \| &\leqslant \| I_N^{1+i\eta} g - I_{\epsilon,N}^{1+i\eta} g \| + \| I_{\epsilon,N}^{i\eta} I_{\epsilon,N}^1 g - I_{\epsilon,N}^{i\eta} I_N^1 g \| \\ &< \frac{\delta}{2} + e^{\pi|\eta|/2} \| I_{\epsilon,N}^1 g - I_N^1 g \| \\ &< \delta, \end{aligned}$$

for $\epsilon > 0$ sufficiently small, and the proof is complete.

Remark. Observe that a slight modification of the above proof yields the *existence* of the boundary group $\{I_N^{i\eta}\}_{\eta\in R}$.

To complete the proof of the regularity of $\{I^\zeta\}_{\zeta\in C^+}$, we turn to the theory of fractional powers of closed operators developed by Balakrishnan [1]. He proves that if A is a closed operator whose resolvent satisfies the property

$$\|\lambda R(\lambda; A)\| \leqslant M, \qquad \text{for all} \quad \lambda > 0,$$

then if $D(A^\infty) = \bigcap_{n\in Z^+} D(A^n)$,

$$\{x \in D(A^\infty) \mid \lambda R(\lambda; A)x \to 0 \text{ as } \lambda \to 0\} \subsetneq \tilde{D}, \tag{4.4.3}$$

where $\tilde{D}$ is the set we defined in Definition 2.1. Of course, if A is the generator of a semi-group of class (C_0), then the set in (4.4.3) is simply equal to $D(A^\infty)$. In general, $\overline{D(A^\infty)} = \overline{D(A)}$ (cf. [1, Lemma 3.1]).

Now consider the integral operator I^1. The resolvent set of I^1 is the left half-plane, since I^1 is the inverse of the differentiation operator D with domain $\{f\in L^p(0, \infty) \mid f$ absolutely continuous on $(0, \infty)$, $f(0) = 0$, $f' \in L^p\}$. In fact, for $\operatorname{Re} \lambda < 0$, and $f\in L^p(0, \infty)$,

$$R(\lambda; D) f(x) = -\int_0^x e^{\lambda(x-t)} f(t)\, dt \qquad (= -W^1_{-\lambda} f(x))$$

(cf. [10, p. 175]).

The operator $-I^1$ satisfies the property that $C^+ \subset \rho(-I^1)$, and for all $f\in L^p(0, \infty)$, $\lambda \in C^+$,

$$\begin{aligned} R(\lambda; -I^1) f(x) &= \frac{1}{\lambda} f(x) - \frac{1}{\lambda^2} \int_0^x e^{(1/\lambda)(t-x)} f(t)\, dt \\ &= \frac{1}{\lambda}\left[f(x) - \frac{1}{\lambda} W^1_{1/\lambda} f(x)\right]. \end{aligned}$$

Thus $\| R(\lambda; -I^1)\| \leqslant 2/\lambda$, since $\|(1/\lambda) W^1_{1/\lambda}\| \leqslant 1$. Thus $-I^1$ satisfies the requirements of Balakrishnan's theory. Moreover, the set in (4.4.3) is equal to $D((-I^1)^\infty)$, since for all $f\in L^p(0, \infty)$ $\|\lambda R(\lambda; -I^1) f\| \to 0$ as $\lambda \to 0$. Indeed, for $\lambda > 0$,

$$\begin{aligned} -\lambda R(\lambda; -I^1) f &= (-D^1)\, R(1/\lambda; -D^1) f \qquad ([10, \text{p. } 177]) \\ &= (1/\lambda)\, R(1/\lambda; -D^1) f - f \\ &\to 0 \qquad \text{as} \quad \lambda \to 0, \end{aligned}$$

because $-D^1$ generates a semi-group of class (C_0). Since $\overline{D((-I^1)^\infty)} = \overline{D(I^1)}$ ([1, Lemma 3.1]), and I^1 is densely defined in $L^p(0, \infty)$, we have the following:

PROPOSITION 4.5. $\bar{\tilde{D}} = L^p(0, \infty)$.

Thus $\{I^\zeta\}_{\zeta\in C^+}$ is regular; by Theorem 2.2, there exists a strongly continuous group of operators $\{I^{i\eta}\}_{\eta\in R}$ on $L^p(0, \infty)$, $1 < p < \infty$, where $I^{i\eta} f = \lim_{N\to\infty} I_N^{i\eta} f$

for each $f \in L^p(0, \infty)$. By Theorem 4.4, $\| I^{i\eta} \| \leqslant e^{\pi|\eta|/2}$, for each $\eta \in R$, and $I^{i\eta} I^{\alpha} = I^{\alpha+i\eta}$ in $L^p(0, \infty)$, for $\alpha > 0$, $\eta \in R$. In fact, in this case it is easy to see that $I^{\alpha} I^{i\eta} = I^{\alpha+i\eta}$, as well. Indeed, $I_N{}^{\zeta} f = P_N I^{\zeta} P_N f$ for $f \in L^p(0, \infty)$, so if $M \leqslant N$, then

$$\begin{aligned} P_M I_N^{i\eta} f &= P_M \lim_{\xi \to 0^+} I_N^{\xi+i\eta} f \\ &= I_M^{i\eta} f. \end{aligned}$$

Therefore if $M \in Z^+$, then $P_M I^{i\eta} f = I_M^{i\eta} f$. If $f \in D(I^{\alpha+i\eta})$, then

$$I^{\alpha+i\eta} f = \lim_{N \to \infty} I_N^{\alpha+i\eta} f$$

and

$$\begin{aligned} I_N I^{i\eta} f &= P_N I^{\alpha} P_N I^{i\eta} f \\ &= I_N{}^{\alpha} I_N^{i\eta} f \\ &= I_N^{\alpha+i\eta} f \to I^{\alpha+i\eta} f. \end{aligned}$$

Thus $I^{i\eta} f \in D(I^{\alpha})$, and $I^{\alpha} I^{i\eta} f = I^{\alpha+i\eta} f$. Applying this argument to $g = I^{-i\eta} f$ yields the desired result.

Now by Theorems 3.3 and 3.4 we have

THEOREM 4.6. (i) *For each* $\eta \in R$,

$$T_{i\eta} \subset I^{-i\eta} M I^{i\eta}.$$

(ii) *If* $\alpha, \beta \in C \backslash \{0\}$ *and* $\operatorname{Re} \alpha = \operatorname{Re} \beta$, *then* T_{α} *is similar to* T_{β} *(as unbounded operators in* $L^p(0, \infty)$*), with* $I^{i \operatorname{Im}(\alpha-\beta)}$ *implementing the similarity.*

Remark. $D(M) \not\subset D(I^1)$.

Now for $f \in L^p(0, \infty)$, $1 < p < \infty$, $\epsilon > 0$ and $\zeta \in C^+$, let

$$J_{\epsilon}{}^{\zeta} f(x) = \frac{1}{\Gamma(\zeta)} \int_x^{\infty} e^{\epsilon(x-t)} (t-x)^{\zeta-1} f(t)\, dt.$$

If we regard $W_{\epsilon}{}^{\zeta}$ as an operator on $L^q(0, \infty)$, with $1/p + 1/q = 1$, then $J_{\epsilon}{}^{\zeta} = (W_{\epsilon}{}^{\zeta})^*$, so that for each $\epsilon > 0$, $\{J_{\epsilon}{}^{\zeta}\}_{\zeta \in C^+}$ is a regular semi-group of bounded linear operators on $L^p(0, \infty)$. By taking P_{ϵ} to be the identity operator on $L^p(0, \infty)$, for each $\epsilon > 0$, we may regard $\{J_{\epsilon}{}^{\zeta}\}_{\zeta \in C^+}$ as the approximating semi-groups of the one-parameter family of operators $\{J^{\zeta}\}_{\zeta \in C^+}$, where J^{ζ} is defined by

$$J^{\zeta} f = \lim_{\epsilon \to 0^+} J_{\epsilon}{}^{\zeta} f,$$

with domain $D(J^{\zeta}) = \{f \in L^p(0, \infty) \mid \lim_{\epsilon \to 0^+} J_\epsilon{}^\zeta f$ exists in $L^p(0, \infty)\}$. In fact, it is easy to see that $\{J^\zeta\}_{\zeta \in C^+}$ is a regular semi-group of unbounded linear operators. Indeed, (2.1.1) follows from Theorem 4.2, and $C_c(0, \infty) \subset \tilde{D}$, so (ii) of Definition 2.1 is trivial. Thus we may conclude the existence of a boundary group $\{J^{i\eta}\}_{\eta \in R}$, where $J^{i\eta} = \lim_{\epsilon \to 0^+} J_\epsilon^{i\eta}$ in the strong operator topology. If M is the multiplication operator, and A the identity operator on $L^p(0, \infty)$, then the results of Theorems 3.3 and 3.4 hold. However, it is now the case that $D(M) \subset D(J^1)$, which we prove in the following lemmas. First, let W^ζ, $\zeta \in C^+$ denote the Weyl fractional integral acting in $L^p(0, \infty)$; i.e.,

$$W^\zeta f(x) = \frac{1}{\Gamma(\zeta)} \int_x^\infty (t - x)^{\zeta - 1} f(t)\, dt,$$

with domain $D(W^\zeta) = \{f \in L^p(0, \infty) \mid W^\zeta f \in L^p(0, \infty)\}$.

LEMMA 4.7. *For each* $n = 0, 1, 2, \ldots,$ *and* $f \in D(W^n)$,

$$J_\epsilon{}^n f = (1 - \epsilon J_\epsilon)^n\, W^n f \qquad (\epsilon > 0). \tag{4.7.1}$$

Proof. We first verify the identity for $n = 1, f \in D(W^1)$. We have (on $(0, \infty)$)

$$\begin{aligned} J_\epsilon f(x) + \epsilon (J_\epsilon W f)(x) &= \int_x^\infty [e^{\epsilon(x-t)} f(t) + \epsilon\, e^{\epsilon(x-t)} (Wf)(t)]\, dt \\ &= \int_x^\infty - \frac{d}{dt} [e^{\epsilon(x-t)} (Wf)(t)]\, dt \\ &= -e^{\epsilon(x-t)}\, Wf(t)|_x^\infty \\ &= (Wf)(x). \end{aligned}$$

Proceeding by induction, suppose (4.7.1) holds for some n. Then if $f \in D(W^{n+1}) \subset D(W^n)$, $W^{n+1} f = W \cdot W^n f$ and

$$\begin{aligned} J_\epsilon^{n+1} f = J_\epsilon J_\epsilon{}^n f &= J_\epsilon (1 - \epsilon J_\epsilon)^n W^n f \\ &= (1 - \epsilon J_\epsilon)^n J_\epsilon W^n f \\ &= (1 - \epsilon J_\epsilon)^n (I - \epsilon J_\epsilon)\, W W^n f \\ &= (1 - \epsilon J_\epsilon)^{n+1} W^{n+1} f. \end{aligned}$$

LEMMA 4.8. *For* $n \in Z^+$, $W^n = J^n$.

Proof. Suppose $f \in D(W^n)$, where $n \in Z^+$. Then by Lemma 4.7, $J_\epsilon{}^n f = (1 - \epsilon J_\epsilon)^n\, W^n f$. However, if D is the differentiation operator with domain

$\{f \in L^p(0, \infty) \mid f$ is absolutely continuous on $(0,\infty)$, $f' \in L^p(0, \infty)\}$, then if $\lambda \in C^+$, $\lambda \in \rho(D)$ and for $f \in L^p(0, \infty)$,

$$R(\lambda; D) f(x) = \int_x^\infty e^{\lambda(x-t)} f(t)\, dt = J_\lambda^1 f(x) \qquad \text{(cf. [10, p. 175]).}$$

Thus $(1 - \epsilon J_\epsilon) f = [1 - \epsilon R(\epsilon; D)] f \to f$ in L^p as $\epsilon \to 0+$, since D generates a semi-group of class (C_0). Also, $\| 1 - \epsilon J_\epsilon \| \leqslant 2$, so that $\lim_{\epsilon \to 0+} J_\epsilon^n f$ exists, and is equal to $W^n f$. Thus $W^n \subset J^n$. On the other hand, suppose $\lim_{\epsilon \to 0+} J_\epsilon^n f$ exists. Then since $(1 - \epsilon J_\epsilon)^n f \to f$, $(1 - \epsilon J_\epsilon)^n f \in D(J^n)$, and $J^n(1 - \epsilon J_\epsilon)^n f = J_\epsilon^n f$, it follows that $f \in D(J^n)$, for J^n is closed. Clearly $J^n f = W^n f$.

LEMMA 4.9. $D(M) \subset D(J^1)$.

Proof. Consider the operators $K_{\eta,\alpha}^-$ defined in [11] by

$$K_{\eta,\alpha}^- f(x) = \frac{1}{\Gamma(\alpha)}\, x^\eta \int_x^\infty (t - x)^{\alpha-1} t^{-\eta-\alpha} f(t)\, dt.$$

Then $K_{0,1}^-$ is a bounded linear operator in $L^p(0, \infty)$ (cf. [11, Theorem 2]), and $J^1 f = K_{0,1}^- M f$, for $f \in D(M)$. Thus $D(M) \subset D(J^1)$, as wanted.

THEOREM 4.10. *If* $\alpha, \beta \in C$ *with* $\operatorname{Re} \alpha = \operatorname{Re} \beta$, *then* $T_\alpha = M + \alpha J$ *is similar to* $T_\beta = M + \beta J$.

5. THE BOUNDARY GROUP OF $\{I^\zeta\}_{\zeta \in C^+}$ AS A WEAK LIMIT

In this section we give another proof of the existence of the boundary group $\{I^{i\eta}\}_{\eta \in R}$; the proof is independent of Theorem 2.2, and only uses some of the results of Section 4. In addition, we obtain the similarity results of Theorem 4.6. The notation will be the same as that in Section 4.

THEOREM 5.1. *As* $\epsilon \to 0^+$, $W_\epsilon^{i\eta}$ *converges to a bounded operator* $I^{i\eta}$ *on* $L^p(0, \infty)$, *in the weak operator topology* (*for each* $\eta \in R$, *and* $1 < p < \infty$). $\{I^{i\eta}\}_{\eta \in R}$ *is a strongly continuous group of operators on* $L^p(0, \infty)$, $\| I^{i\eta} \| \leqslant e^{\pi|\eta|/2}$, and

$$I^{i\eta} I^\xi = I^\xi I^{i\eta} = I^{\xi+i\eta} \qquad (\xi > 0,\ \eta \in R)$$

on $D(I^\xi) = \{f \in L^p(0, \infty) \mid I^\xi f \in L^p(0, \infty)\} = D(I^{\xi+i\eta})$.

Proof. Fix $\eta \in R$ and $f \in L^p(0, \infty)$. By Theorem 4.2, $\| W_\epsilon^{i\eta} f \|_p$ is bounded with respect to $\epsilon > 0$. Since $L^p(0, \infty)$ is reflexive for $1 < p < \infty$, $\{W_\epsilon^{i\eta} \mid \epsilon > 0\}$ has weak limit points in $L^p(0, \infty)$. Let g be such a limit point.

For $\xi > 0$, the boundary group $W_\epsilon^{i\eta}$ satisfies

$$(W_\epsilon^{\,\xi} W_\epsilon^{i\eta} f)(x) = W_\epsilon^{\xi+i\eta} f(x) = \frac{1}{\Gamma(\xi + i\eta)} \int_0^x e^{\epsilon(t-x)} (x - t)^{\xi+i\eta-1} f(t)\, dt.$$

The difference between the expression on the right and $I^{\xi+i\eta}f(x)$ has modulus no larger than

$$\frac{1}{|\Gamma(\xi+i\eta)|}\int_0^x (1-e^{\epsilon(t-x)})(x-t)^{\xi-1}\,|f(t)|\,dt$$
$$\leqslant \frac{1}{|\Gamma(\xi+i\eta)|}\,\epsilon\int_0^x (x-t)^{\xi}\,|f(t)|\,dt \to 0 \qquad \text{as} \quad \epsilon\to 0,$$

since $0\leqslant 1-e^{\epsilon(t-x)}\leqslant \epsilon(x-t)$. Therefore, when $\epsilon'\to 0^+$ through a sequence such that $W_{\epsilon'}^{i\eta}f\to g$ weakly,

$$W_{\epsilon'}^{\xi}W_{\epsilon'}^{i\eta}f(x)\to I^{\xi+i\eta}f(x). \tag{5.1.1}$$

On the other hand, the left side of (5.1.1) can be written as

$$\frac{1}{\Gamma(\xi)}\int_0^x (x-t)^{\xi-1}W_{\epsilon'}^{i\eta}f(t)\,dt + R_{\epsilon'}\,, \tag{5.1.2}$$

where one has

$$|R_{\epsilon'}|\leqslant \frac{\epsilon'}{\Gamma(\xi)}\int_0^x (x-t)^{\xi}\,|W_{\epsilon'}^{i\eta}f(t)|\,dt$$
$$\leqslant \frac{\epsilon'}{\Gamma(\xi)}\,\|(x-t)^{\xi}\|_{L^q(0,x)}\,\|W_{\epsilon'}^{i\eta}f\|_{L^p(0,x)} = O(\epsilon'),$$

(where $1/p+1/q=1$) by (4.2.1). The first term in (5.1.2) converges to $I^{\xi}g(x)$ as $\epsilon'\to 0$ if $\xi>1/p$ (this ensures that $(x-t)^{\xi-1}\in L^q(0,x)$ for each fixed x). Consequently, we obtain

$$I^{\xi}g = I^{\xi+i\eta}f \qquad (\eta\in R,\ \xi>1/p).$$

For $\xi>0$ arbitrary, $1+\xi>1>1/p$, and therefore

$$I^1(I^{\xi}g) = I^{1+\xi}g = I^{1+\xi+i\eta}f = I^1(I^{\xi+i\eta}f).$$

Since I^1 is one-to-one on the locally integrable functions, it follows that

$$I^{\xi}g = I^{\xi+i\eta}f \qquad (\xi>0,\ \eta\in R). \tag{5.1.3}$$

Suppose $h\in L^p(0,\infty)$ is also a weak limit point of $\{W_{\epsilon}^{i\eta}\mid \epsilon>0\}$. Taking $\xi=1$ in (5.1.3) (and the corresponding relation for h), one obtains

$$I^1h = I^{1+i\eta}f = I^1g$$

and therefore $h=g$ as elements of $L^p(0,\infty)$. This proves the *existence* of the weak limit of $W_{\epsilon}^{i\eta}f$ as $\epsilon\to 0^+$. We denote it by $I^{i\eta}f$. Relation (5.1.3) can be written

$$I^{\xi}I^{i\eta}f = I^{\xi+i\eta}f \qquad (\xi>0,\ \eta\in R), \tag{5.1.4}$$

valid for all $f \in L^p(0, \infty)$. It follows from Theorem 4.2 that

$$\| I^{i\eta} \| \leqslant e^{\pi|\eta|/2} \qquad (\eta \in R).$$

Next, suppose $f \in D(I^\xi) = \{f \in L^p(0, \infty) \mid I^\xi f \in L^p(0, \infty)\}$. Then $I^{i\eta}(I^\xi f) \in L^p(0, \infty)$, and by (5.1.4) and Fubini's theorem

$$I^1 I^{i\eta}(I^\xi f) = I^{1+i\eta}(I^\xi f) = I^{1+\xi+i\eta} f = I^1(I^{\xi+i\eta} f).$$

Hence $I^{i\eta}(I^\xi f) = I^{\xi+i\eta} f$, as elements of $L^p(0, \infty)$. In particular, $I^{\xi+i\eta} f \in L^p(0, \infty)$; i.e., $D(I^\xi) \subset D(I^{\xi+i\eta})$. The same argument with $I^{-i\eta}$ and $I^{\xi+i\eta}$ gives the reverse inclusion. This proves that $D(I^\xi) = D(I^{\xi+i\eta})$, and on this domain, $I^\xi I^{i\eta} = I^{i\eta} I^\xi = I^{\xi+i\eta}$.

The group property of $I^{i\eta}$ can be proved as follows. For each $f \in L^p(0, \infty)$ and $\eta, \tau \in R$, we have by (5.1.4)

$$\begin{aligned} I^2 I^{i(\eta+\tau)} f &= I^{(1+i\eta)+(1+i\tau)} f = I^{1+i\eta} I^{1+i\tau} f \\ &= I^{1+i\eta} I^1 I^{i\tau} f = I^1 I^{1+\eta} I^{i\tau} f \\ &= I^2 I^{i\eta} I^{i\tau} f, \end{aligned}$$

and therefore $I^{i(\eta+\tau)} = I^{i\eta} I^{i\tau}$ since I^2 is one-to-one.

For each fixed $f \in L^p(0, \infty)$, $I^{i\eta} f$ is a weakly measurable function with values in $L^p(0, \infty)$, as the weak limit of the strongly continuous functions $W_\epsilon^{i\eta} f$ (as $\epsilon \to 0^+$). Since $L^p(0, \infty)$ is separable, $I^{i\eta} f$ is strongly measurable (cf. Corollary 2 of Theorem 3.5.3 in [5]). Due to the group property, it now follows from Theorem 10.2.3 in [5], that $I^{i\eta}$ is strongly continuous on R, for each $f \in L^p(0, \infty)$. This completes the proof.

Next we discuss the similarity results.

LEMMA 5.2. *Let $\xi > 0$, and let n be the first integer $\geqslant \xi$. Then for each $f \in D(I^n)$, $I^{\xi+i\eta} f$ is the weak limit of $W_\epsilon^{\xi+i\eta} f$ as $\epsilon \to 0^+$ ($\eta \in R$).*

Proof. As in the proof of Lemma 4.7, one sees easily that for each $k = 0, 1, 2,\ldots$ and each locally integrable f,

$$W_\epsilon^k f = (1 - \epsilon W_\epsilon)^k I^k f, \qquad \epsilon > 0.$$

Thus for $f \in D(I^k)$,

$$\begin{aligned} \| W_\epsilon^{k+i\eta} f \| &= \| W_\epsilon^{i\eta} W_\epsilon^k f \| = \| W_\epsilon^{i\eta}(1 - \epsilon W_\epsilon)^k I^k f \| \\ &\leqslant \| W_\epsilon^{i\eta} \| \, \| 1 - \epsilon W_\epsilon \|^k \, \| I^k f \| \\ &\leqslant e^{\pi|\eta|/2}(1 + \| \epsilon W_\epsilon \|)^k \, \| I^k f \| \\ &= 2^k e^{\pi|\eta|/2} \| I^k f \|, \qquad k = 0\ 1, 2,\ldots . \end{aligned}$$

Applying the three lines theorem to the function

$$\Phi(\zeta) = e^{\pi\zeta^2/2} W_\epsilon^\zeta f$$

in the strip $n - 1 \leqslant \xi \leqslant n$, one obtains

$$\| W_\epsilon^{\xi+i\eta} f \| \leqslant e^{\pi/4} 2^\xi e^{\pi|\eta|/2} \| I^n f \|^{\xi-n+1} \| I^{n-1} f \|^{n-\xi}$$

for $n - 1 \leqslant \xi \leqslant n$ and $f \in D(I^n)$ ($\subset D(I^{n-1})$). In particular, $\| W_\epsilon^{\xi+i\eta} f \|$ is bounded with respect to $\epsilon > 0$, for fixed $f \in D(I^n)$. Let g be a weak limit point of $\{W_\epsilon^{\xi+i\eta} f \mid \epsilon > 0\}$ (with ξ, η, and f fixed as above), say $g = \text{weak} \lim_{\epsilon' \to 0+} W_{\epsilon'}^{\xi+i\eta} f$. For each fixed $x \geqslant 0$, the characteristic function $\chi_{[0,x]} \in L^q(0, \infty)$, hence

$$I^1 g(x) = \lim_{\epsilon' \to 0+} \int_0^x W_{\epsilon'}^{\xi+i\eta} f(t)\, dt.$$

By (5.1.1), the integrand converges pointwise to $I^{\xi+i\eta} f$, and is dominated by

$$\frac{\Gamma(\xi)}{|\Gamma(\xi + i\eta)|} I^\xi |f| (t) \in L^1(0, x).$$

By dominated convergence, it follows that

$$I^1 g = I^1 I^{\xi+i\eta} f,$$

hence $g = I^{\xi+i\eta} f$, and the lemma follows.

LEMMA 5.3. *M is a "closed" operator with respect to weak sequential convergence in $L^p(0, \infty)$; that is, if $\{f_n\} \subset D(M)$, $f_n \to^w f$ and $Mf_n \to^w g$, then $f \in D(M)$ and $Mf = g$.*

Proof. For all h in the domain of M in $L^q(0, \infty)$,

$$\int_0^\infty h(x)\, g(x)\, dx = \lim_{n\to\infty} \int_0^\infty h(x)\, Mf_n(x)\, dx = \lim_{n\to\infty} \int_0^\infty Mh(x)\, f_n(x)\, dx$$
$$= \int_0^\infty Mh(x)\, f(x)\, dx = \int_0^\infty h(x)\, Mf(x)\, dx.$$

Since $D(M)$ in $L^q(0, \infty)$ is dense in $L^q(0, \infty)$, one has in particular $f \in D(M) = \{f \in L^p(0, \infty) \mid Mf \in L^p(0, \infty)\}$, and $Mf = g$.

LEMMA 5.4. *$[M, W_\epsilon^\zeta] = \zeta W_\epsilon^{\zeta+1}$ on $D(M)$, for all $\zeta \in \bar{C}^+$, $\epsilon > 0$.*

Proof. For Re $\zeta > 0$, this is verified by direct computation. In particular,

$W_\epsilon^\zeta D(M) \subset D(M)$ for all $\epsilon > 0$ and $\zeta \in C^+$. Fix $f \in D(M)$, $\epsilon > 0$ and $\eta \in R$; for all $\xi > 0$, $W_\epsilon^{\xi+i\eta} f \in D(M)$, $W_\epsilon^{\xi+i\eta} f \to W_\epsilon^{i\eta} f$ as $\xi \to 0^+$, and

$$\begin{aligned} M(W_\epsilon^{\xi+i\eta} f) &= W_\epsilon^{\xi+i\eta}(Mf) + (\xi + i\eta)\, W_\epsilon^{\xi+i\eta+1} f \\ &\to W_\epsilon^{i\eta} Mf + i\eta W_\epsilon^{i\eta+1} f \qquad \text{as} \quad \xi \to 0^+. \end{aligned}$$

Since M is closed (with respect to the usual Banach space structure of $L^p(0, \infty)$), it follows that $W_\epsilon^{i\eta} f \in D(M)$ and $M W_\epsilon^{i\eta} f = W_\epsilon^{i\eta} Mf + i\eta W_\epsilon^{i\eta+1} f$.

THEOREM 5.5. *$[M, I^{i\eta}] f = i\eta I^{i\eta+1} f$, for all $f \in D(M) \cap D(I^1)$.*

Note. $D(M) \cap D(I^1)$ is the maximal domain on which such an identity can possibly hold.

Proof. Fix $f \in D(M) \cap D(I)$, and $\eta \in R$. By Lemma 5.4,

$$M W_\epsilon^{i\eta} f = W_\epsilon^{i\eta} Mf + i\eta W_\epsilon^{i\eta+1} f, \qquad \epsilon > 0. \tag{5.5.1}$$

When $\epsilon \to 0^+$, $W_\epsilon^{i\eta} f \to^w I^{i\eta} f$ (Theorem 5.1), and by (5.5.1), Theorem 5.1 and Lemma 5.2, $M W_\epsilon^{i\eta} f \to^w I^{i\eta} Mf + i\eta I^{i\eta+1} f$. By Lemma 5.3, it follows that $I^{i\eta} f \in D(M)$ and $M I^{i\eta} f = I^{i\eta} Mf + i\eta I^{i\eta+1} f$, as wanted.

COROLLARY 5.6. *Let $\mu, \nu \in C\backslash\{0\}$ with $\operatorname{Re} \mu = \operatorname{Re} \nu$. Then $M + \mu I^1$ is similar to $M + \nu I^1$. That is,*

$$(M + \mu I^1) I^{i\eta} = I^{i\eta}(M + \nu I^1)$$

where $\eta = \operatorname{Im}(\nu - \mu)$.

Proof. The proof is similar to that of Theorem 3.4, and we omit the details.

Remark. The boundary group constructed in Theorem 5.1 is the same as that obtained in Theorem 2.2, since for $I^1 f$ with $f \in D(I^1)$, both are equal to $I^{1+i\eta} f$, and Range (I^1) is dense in $L^p(0, \infty)$.

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